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Fractional quantum Hall effect on the 2-sphere: a case study of symmetry reduction

Roger Sollie† and Kåre Olaussen

Gruppe for teoretisk fysikk, Universitetet i Trondheim, Norges Tekniske Høgskole, N-7034 Trondheim-NTH, Norway

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Abstract. We make a case study of how a standard class of Hamiltonians for the fractional quantum Hall effect (modelled in a spherical geometry) can be explicitly symmetry reduced with respect to the full rotation group, and perform finite-size calculations for small particle numbers.

1. Introduction

The discovery of the fractional quantum Hall effect (FQHE) [1] has revealed a new interesting many body effect. At high magnetic field ($B \simeq 5$-30 T) and low temperatures ($T < 1$ K), a two-dimensional system of electrons may exhibit plateaux in the transversal conductivity σ_{xy}, accompanied by deep minima in the longitudinal conductivity σ_{xx}. The plateaux are quantised to conductivities $\sigma_{xy} = \nu e^2/h$, with ν a rational number, and occur for fillings of the Landau levels around the corresponding fractions ν. Thus far one has observed plateaux, or precursive behaviour for development of plateaux, at fractions $\nu = 1/3, 2/3$ [1], $4/3, 5/3$ [2], $7/3, 8/3$ [3], $1/5$ [4], $2/5, 3/5, 4/5$ [2], $7/5, 8/5$ [5], $1/7$ [6], $2/7$ [2], $3/7, 4/7$ [7], $9/7, 10/7, 11/7$ [5], $19/7$ [8], $2/9$ [9], $4/9, 5/9$ [7], $13/9$ [5], $2/11, 3/11$ [9], $5/11, 6/11, 6/13, 7/13$, and $5/2$ [8]. The effect is only observed in very pure and high mobility samples ($\mu \simeq 0.1$-$7 \times 10^6\ \mathrm{cm^2\,V^{-1}\,s^{-1}}$), most commonly with the two-dimensional electron system being realised in an $Al_{1-x}Ga_xAs$-GaAs heterojunction, but also in Si-MOSFETs [10]. Also, systems with hole carriers have been observed [2].

The constancy of σ_{xy} with the change of filling factor suggests an extraordinary stability of the ground state at the particular fractional fillings. Laughlin [11] has interpreted the ground state at filling factor $\nu = 1/m$, $m = 3, 5, \ldots$, as an incompressible liquid of highly correlated spin-polarised electrons in the lowest Landau level, with the elementary excitations of the system being *fractionally charged* quasiparticles (with charge $e^* = \pm\nu e$), separated from the ground state by energy gaps $\Delta_\pm$. This implies that at these fillings there are cusps in the energy per particle, $\varepsilon(\nu) \equiv E(N, B)/N$, as functions of the filling factor,

$$\left(\frac{\partial\varepsilon}{\partial\nu}\right)_{\nu_+} - \left(\frac{\partial\varepsilon}{\partial\nu}\right)_{\nu_-} = \frac{1}{\nu^2}(\Delta_+ + \Delta_-)$$

which in turn explains the stability of the ground state. Moreover, the very precise

† Present address: Department of Physics, Temple University, Philadelphia, PA 19122, USA.

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quantisation of $h\sigma_{xy}/e^2$ to the rational values ν (observed accurate to 3 parts in 10^5 for the 1/3 and 2/3 fillings) is explained as a consequence of the fractional charge of the quasiparticles. The plateaux observed at the more complicated fractions is explained as a hierarchy where the quasiparticles of one generation condense into new incompressible states similar to the primary Laughlin states, and in turn create a new generation of quasiparticles [12]. To date all observed fractions except the one at $\nu = 5/2$ (which is believed to be a spin-unpolarised state [13]) are in agreement with the hierarchy picture, although there is now good evidence that some states with $1 < \nu < 2$ may undergo phase transitions from fully spin-polarised to spin-unpolarised states [14]. Also, the fractional charge of the quasiparticle excitations may have been observed [15] to be in agreement with the hierarchy picture.

From a theorist's point of view the FQHE represents a beautiful problem. At the outset the model is essentially two dimensional, since the motion transverse to the electron layer is quantised with an energy gap $\Delta E \simeq 20$–50 meV$\leftrightarrow$250–600 K, and thus essentially frozen out at the temperatures $T \leqslant 1$ K which are actual for observing the effect. Also, at the relevant magnetic fields the gap between two successive Landau levels, $\hbar\omega_c \simeq 10$–50 meV$\leftrightarrow$120–600 K, is sufficiently large that only the states in the lowest unfilled Landau level are important. Finally, only the fully spin-polarised states in this Landau level are believed to be essential for a qualitative explanation of the effect. Quantitatively this approximation is considerably more doubtful, however, since the spin-reversed states are separated by a gap $\Delta_s \simeq 0.2$–1 meV$\leftrightarrow$2–10 K which may be comparable to the intralevel Coulomb interaction energies $\Delta_c \leqslant 4$ K. In any case, by projecting the model onto the lowest unfilled Landau level the model becomes essentially one dimensional (although non-homogeneous, and with a non-local interaction) and dependent on only one relevant parameter, the filling factor ν. This makes the model sufficiently simple that it may serve as a standard reference point, and inspires a thorough analysis of it.

Despite the simplicity of formulation, and of parameter space, the model has an extremely rich and interesting structure, exemplified by the exotic concepts as topological quantisation [16], charge fractionalisation [11] and fractional statistics [17] which have been associated with it.

At the same time, finite-particle versions of the model can be viewed simply as a plain numerical problem of diagonalising finite-dimensional matrices. It can be used as a theoretical laboratory where approximation methods and numerical calculation techniques can be tested. The approach taken to the problem in this paper is mainly from the latter point of view.

The ground-state properties, such as the ground-state energy per particle, of the electron gas can be studied by calculations of a small number of particles. Also the full excitation spectrum can be found. Finite-size calculations with a small number of electrons have been performed (for further references see the book edited by Prange and Girvin [18] and by Chakraborty and Pietiläinen [19]). This is most easily done by mapping the problem onto the surface of a sphere [20, 21], with the perpendicular magnetic field created by a huge magnetic monopole at the centre of the sphere. The advantage of this construction is that one need not choose any boundary conditions, as one has to when the electrons are confined to a 2D Euclidean plane. Also, the commonly used periodic boundary conditions are less convenient when a magnetic field is present, basicly due to the more complex topology of a torus compared with the surface of a sphere, and the fact that translations in the x and y directions no longer commute with each other.

The limitations of finite-size calculations are essentially set by the dimension of the Hamiltonian matrix one has to diagonalise. For N particles at filling factor ν the full dimension of the Hilbert space is $\binom{N/\nu}{N}$. However, by straightforward use of the fact that J_z is a good quantum number, the Hamiltonian decomposes into (configuration interaction) matrices whose dimension behaves like [22]

$$C(N, \nu, J_z = 0) \simeq [\sqrt{3}\,\nu/\pi(1-\nu)N^2] \exp\{-(N/\nu)[\nu \ln \nu + (1-\nu)\ln(1-\nu)]\}$$

for large N. This dimension increases very rapidly with the particle number, limiting N to about 10–12. However, since J is also a good quantum number, one may try to utilise this fact to further reduce the dimension of the eigenvalue problems. For the fully rotationally symmetric states this leads to a reduction in the linear dimensions of the matrices by a factor which behaves like [22]

$$D/C \simeq 6\nu^2/[(1-\nu)N^3]$$

for large N. Here D is the dimension of the $J=0$ Hamiltonian matrices. But it is non-trivial to construct the latter matrices in an effective way. In fact, one might fear that the numerical work required to *construct* the symmetry reduced matrices exceeds the savings in diagonalising them. The main purpose of this work is to make a case study of this symmetry reduction up to the (rather limited) case of $N=4$ particles, relying heavily on analytic methods. Our conclusion is that there are definite savings in the numerical work required, making it possible to cover a region of parameter (N, ν) space which would be very difficult to reach, even with the largest supercomputers, when working with the much larger configuration interaction matrices.

The rest of this paper is organised as follows. In section 2 we define the model and show that regarding energy, many potentials are irrelevant when the Hamiltonian is projected onto the lowest Landau level. This is basically due to the Pauli principle. In sections 3 and 4 we consider the two- and three-particle cases for arbitrary potentials. In section 5 we show how one can construct the minimal Hamiltonian matrix for four particles with $J=0$ and set the stage for the numerical work. This is discussed in section 6 along with some comments on our results. In appendix 1 we derive the transformation between two equivalent ways of expressing the Hamiltonian in terms of two-fermion operators. To evaluate matrix elements the commutator algebra of the different two-fermion operators is useful. This we have derived in appendix 2. In appendix 3 we explicitly evaluate the matrix elements required to construct the Hamiltonian matrix.

2. The model

The model system we want to study consists of N electrons confined to the surface of a 2-sphere of radius R. The electrons move in a magnetic field B that is homogeneous on the sphere, originating from a magnetic monopole with $q = 2\pi R^2 B/\Phi_0$ units of magnetic charge located at the centre of the sphere. Here $\Phi_0 = h/e$ is the elementary flux quantum. In second quantised language the pair interaction operator is

$$V = \tfrac{1}{2} \int \mathrm{d}^2\Omega\, \mathrm{d}^2\Omega' :\rho(\Omega)\, V(\Omega, \Omega')\rho(\Omega'): \tag{2.1}$$

where the number density operator is related to the fermion field operators as $\rho(\Omega) = \Psi^+(\Omega)\Psi(\Omega)$. We shall assume that the electrons interact via a rotational invariant

potential, in which case it can be expanded in spherical harmonics

$$V(\Omega,\Omega')=V(\cos\vartheta)=\sum_L v_L P_L(\cos\vartheta)=\sum_{LM} v_L\frac{4\pi}{2L+1}\,Y^*_{L,M}(\Omega)\,Y_{L,M}(\Omega') \tag{2.2}$$

with ϑ the spherical angle between Ω and Ω'.

When the magnetic field is sufficiently high we may simplify the model by projecting the Hamiltonian onto the fully spin-polarised states of the lowest Landau level†. The field operators $\Psi(\Omega)$ can then be expanded as

$$\Psi(\Omega)=\sum_{m=-q}^{q} a_m Y^{(-q)}_{qm}(\Omega) \tag{2.3}$$

where $Y^{(-q)}_{jm}(\Omega)$ are the monopole harmonics of Wu and Yang [23], and a_m is the annihilation operator for an electron with angular momentum $J_z=\hbar m$. Since

$$Y^{(-q)}_{qm}(\theta,\varphi)=\left(\frac{2q+1}{4\pi}\right)^{1/2}\binom{2q}{q+m}^{1/2}\cos^{q+m}(\tfrac{1}{2}\theta)\sin^{q-m}(\tfrac{1}{2}\theta)\,e^{i(m\mp q)\varphi} \tag{2.4}$$

(in a gauge patch which is regular on the Northern/Southern hemisphere) this electron will as $q\to\infty$ be localised to the region where $\cos\theta\simeq m/q$.

Inserting the expansions (2.2) and (2.3) into equation (2.1) we may carry out the angular integrations. The integrals over products of three (monopole) spherical harmonics [24] ($Y^{(q)}_{qm}Y^{(-q)}_{qn}Y_{LM}$) leads to the expression

$$V=\sum_L\frac{\alpha_L}{\sqrt{2L+1}}\sum_{M=-L}^{L}(-1)^M:C_{LM}C_{L,-M}: \tag{2.5}$$

where α_L is related to the original expansion coefficient v_L by

$$\alpha_L=\tfrac{1}{2}\sqrt{2L+1}\binom{4q+1}{2q}^{-1}\binom{4q+1}{2q-L}\frac{2q+1}{2L+1}\,v_L \tag{2.6}$$

and the C_{LM} are spin-L spherical tensors,

$$\begin{aligned}C_{LM}&=\sum_{m,n}(-1)^{n+q}\langle qqmn|LM\rangle a^\dagger_m a_{-n}\\&=\sum_{m,n}(-1)^{M-n+q}\sqrt{2L+1}\begin{pmatrix}q & q & L\\ m & -n & -M\end{pmatrix}a^\dagger_m a_n\end{aligned} \tag{2.7}$$

with $\begin{pmatrix}\cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot\end{pmatrix}$ a Wigner 3-j symbol. Notice that the factor $\binom{4q+1}{2q}^{-1}\binom{4q+1}{2q-L}$ in (2.6) has the effect of damping out the large-L components of the interaction potential. For large q this factor behaves like $\exp(-L^2/q)$. This will essentially damp out all the components with $L\geqslant q^{1/2}=R/l_B$, where $l_B=(\hbar/eB)^{1/2}$ is the magnetic length. This damping is due to the fact that states in the lowest Landau level can only be localised to a region of linear extension l_B. Thus, variations in the pair potential on shorter length scales will be averaged to zero.

Due to this damping we may restrict L to the region $L\ll q$ as $q\to\infty$. Therefore, the asymptotic relation [25]

$$(-1)^{M-n+q}\sqrt{2L+1}\begin{pmatrix}q & q & L\\ m & -n & -M\end{pmatrix}\sim\sqrt{\frac{4\pi}{2q+1}}\,P_{LM}(x)$$

† As discussed in the introduction, with the field strengths used in experiments this may be a doubtful approximation for the spin polarisation.

is valid in this limit. Here $P_{LM}(x) = Y_{LM}(\theta, \phi = 0)$, with $x \equiv \cos\theta \equiv (M+2n)/2q$. Thus, we may approximate

$$C_{LM} \simeq \sqrt{\frac{4\pi}{2q+1}} \sum_n P_{LM}(x) a^\dagger_{n+M} a_n \tag{2.8}$$

as $q \to \infty$.

The interaction (2.5) can be rewritten in terms of pair creation and annihilation operators, defined as

$$\begin{aligned} A^\dagger_{LM} &= \sum_{m,n} \langle qqmn|LM\rangle a^\dagger_m a^\dagger_n = \sum_{m,n} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a^\dagger_m a^\dagger_n \\ A_{LM} &= \sum_{m,n} \langle qqmn|LM\rangle a_n a_m = \sum_{m,n} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a_n a_m . \end{aligned} \tag{2.9}$$

These operators are by construction spherical tensors of rank L, as can also be verified directly from their commutation rules with the angular momentum operators,

$$\begin{aligned} [J_z, A^\dagger_{LM}] &= M A^\dagger_{LM} \\ [J_\pm, A^\dagger_{LM}] &= \sqrt{(L \mp M)(L \pm M + 1)}\, A^\dagger_{L,M\pm 1}. \end{aligned} \tag{2.10}$$

Using the symmetry properties of the 3-j symbols, and the anticommutation rule for the fermion operators, we find

$$\begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a_m a_n = -(-1)^{2q+L} \begin{pmatrix} q & q & L \\ n & m & -M \end{pmatrix} a_m a_n$$

which implies that the $A_{L,M}$ are non-zero only for L such that $(-1)^{2q+L} = -1$ (i.e. odd L when q is integer and even L when q is half interger).

To express the interaction operator in terms of $A^\dagger_{LM}$ and A_{LM} we define

$$\begin{aligned} V_L &= \frac{1}{\sqrt{2L+1}} \sum_M (-1)^M :C_{LM} C_{L,-M}: \\ W_L &= -\frac{1}{\sqrt{2L+1}} \sum_M A^\dagger_{LM} A_{LM}. \end{aligned} \tag{2.11}$$

As shown in appendix 1 we then have the relations

$$V_L = \sum_J R_{LJ} W_J \qquad W_L = \sum_J R_{LJ} V_J \tag{2.12}$$

where the transformation matrix R_{LJ} is given by

$$R_{LJ} = \sqrt{(2L+1)(2J+1)} \begin{Bmatrix} q & q & L \\ q & q & J \end{Bmatrix} \tag{2.13}$$

with the object in curly braces a Wigner 6-j symbol. From the properties of the 6-j symbols it follows that R_{LJ} is a real, symmetric and orthogonal matrix. We thus have two alternative ways of representing the interaction (2.5)

$$V = \sum_L \alpha_L V_L = \sum_J \beta_J W_J \tag{2.14}$$

with $\alpha_L = \Sigma_J R_{LJ} \beta_J$, $\beta_L = \Sigma_J R_{LJ} \alpha_J$.

In addition to the electron-electron interaction we shall assume the presence of a homogeneous neutralising background, of density $N/4\pi R^2$, where N is the number

of electrons. The electron-background and the background-background interaction then gives an additional contribution to the Hamiltonian. This becomes $-\frac{1}{2}v_0\boldsymbol{N}^2$ with $\boldsymbol{N}=\Sigma_n a_n^\dagger a_n$ the number operator. There is also another contribution to the Hamiltonian, coming from the kinetic energy of the electrons. This is equal to $\frac{1}{2}\hbar\omega_c\boldsymbol{N}=(\hbar^2/2m^*l_B^2)\boldsymbol{N}$ for electrons in the lowest Landau level (with m^* the effective mass of the electrons), but following similar treatments we shall not include it in our Hamiltonian. Thus, the model Hamiltonian we choose is

$$\boldsymbol{H}=\boldsymbol{V}-\tfrac{1}{2}v_0\boldsymbol{N}^2. \tag{2.15}$$

The canonical choice for interaction potential is the (3D) Coulomb interaction $V(r)=e^2/4\pi\epsilon r$. Taking r to be the chord distance between two points on the sphere, $r=2\sin(\vartheta/2)R$, this is equivalent to taking $v_L=e^2/4\pi\epsilon l_B q^{1/2}$.

Since the operators $A_{LM}^\dagger$ and A_{LM} are identically zero when $2q+L$ is an even integer, the same necessarily holds for the W_L. This means that only $[q+\frac{1}{2}]$ of the $2q+1$ potential operators V_L are linearly independent. Any combination $\Sigma_L\,\alpha_L'V_L$, where $\alpha_L'=\Sigma_J'\,R_{LJ}\beta_J$ (with the sum restricted to such J that $2q+J$ is even), is identically zero. Thus, there is a large class of irrelevant pair potentials which, as a consequence of the Pauli principle, leads to no interaction at all when the model is projected onto the lowest Landau level. The most obvious of these is the point interaction, $V(\Omega,\Omega')=\delta(\Omega,\Omega')$. However, the irrelevant pair potentials behave in general rather pathologically.

3. The two-particle states

To obtain a physical interpretation of the various quantities involved it is useful first to consider the case of two particles. Let $|0\rangle$ denote the zero-particle state, defined such that $a_m|0\rangle=0$ for all $m=-q,\,-q+1,\ldots,q$. We can apply the operator $A_{L,M}^\dagger$ to create a two-particle state. The commutation rules (2.10) implies that this state has total angular momentum $\boldsymbol{J}^2=L(L+1)$, and $J_z=M$. The normalised state is

$$|L,M\rangle=\frac{1}{\sqrt{2}}A_{L,M}^\dagger|0\rangle. \tag{3.1}$$

Since L and M are good quantum numbers, and since this is the only two-particle state with these quantum numbers, it must be an eigenstate of the Hamiltonian. Furthermore, the energy is independent of M:

$$E_L^{(2)}=\langle L,M|\boldsymbol{V}|L,M\rangle=-\frac{1}{2}\sum_{J,J_z}\frac{\beta_J}{\sqrt{2J+1}}\langle 0|A_{L,M}A_{J,J_z}^\dagger A_{J,J_z}A_{L,M}^\dagger|0\rangle=-\frac{2\beta_L}{\sqrt{2L+1}} \tag{3.2}$$

where the matrix element is calculated by use of the commutation rules in appendix 2.

As $q\to\infty$ this expression should be related to the classical energy between two particles on the sphere. In this limit a state with angular momentum L must be a superposition of configurations with constant spherical angle ϑ_L between the two particles. Here ϑ_L must be the angle between two vectors $\boldsymbol{q}$ that are coupled so that $\boldsymbol{L}=\boldsymbol{q}+\boldsymbol{q}$, i.e.

$$\cos\vartheta_L=\frac{L(L+1)-2q(q+1)}{2q(q+1)}. \tag{3.3}$$

We thus expect the two-particle energy to approach a classical value

$$E_L^{(2)} \simeq V(\cos\vartheta_L) = \sum_J v_J P_J(\cos\vartheta_L) \tag{3.4}$$

as q, $L \to \infty$. Comparing this expression with equations (3.2) and (2.13), we must have

$$\beta_L = \frac{1}{2}\sum_J \frac{2q+1}{\sqrt{2J+1}} R_{LJ} v_J \simeq -\tfrac{1}{2}\sqrt{2L+1}\sum_J v_J P_J(\cos\vartheta_L)$$

as q, $L \to \infty$ (we have assumed that the potential behaves smoothly on the scale l_B, so that we may set $e^{-J^2/q} \simeq 1$ in the definition of α_J). Thus, we should have $R_{LJ} \simeq [(2L+1)(2J+1)]^{1/2} P_J(\cos\vartheta_L)/(2q+1)$ as q, $L \to \infty$. Indeed, this relation follows from the asymptotic behaviour of the 6-j symbols [25], with precisely the definition (3.3) for $\cos\vartheta_L$. The quality of the asymptotic relation (3.4) is tested in figure 1 for the Coulomb potential when $2q+1 = 100$.

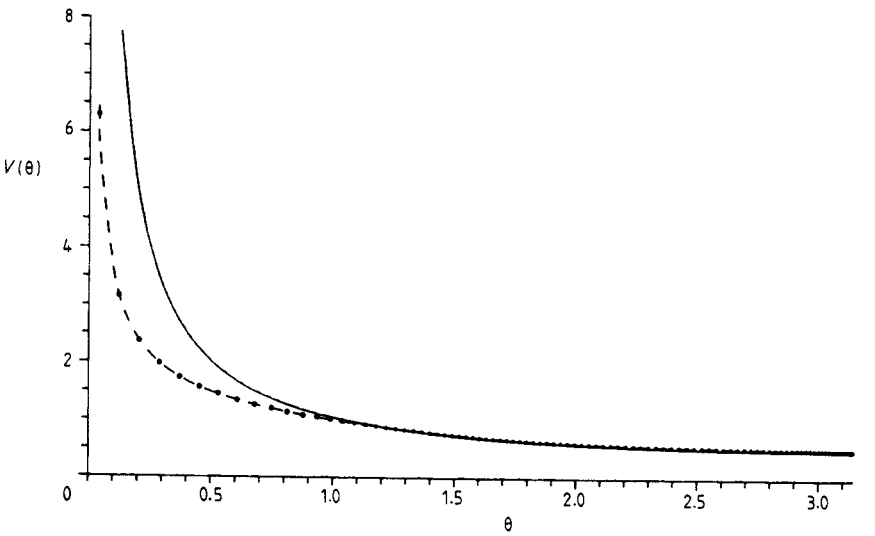

Figure 1. The effective two-particle potential as a function of the spherical angle ϑ. The full curve is the classical potential. The broken curve is the energy as calculated by (3.2), and with the angle calculated by (3.3).

The quantities $E_L^{(2)} = -2\beta_L/\sqrt{2L+1}$ are related to the pseudopotential coefficients V_m of Haldane and Rezayi [20] by $E_L^{(2)} = V_{2q-L}$.

From the relation $\beta_L \simeq -\frac{1}{2}\sqrt{2L+1}\, V(\cos\vartheta_L)$ it follows that $V(\cos\vartheta_L)$ will oscillate wildly if it is an irrelevant potential (so that every second β_L becomes zero). The presence of the smoothing factor $\exp(-J^2/q)$ in the definition of α_J requires the irrelevant potentials to be even more pathological.

4. The three-particle Laughlin states

The spherical analogues of the Laughlin-Jastrow (LJ) wavefunctions exist for the combinations $q = \frac{1}{2}m(N-1)$ of magnetic charge q and particle number N [20]. Here

m is an odd integer. The corresponding filling factor is

$$\nu \equiv \frac{N}{2q+1} = \frac{m^{-1}}{1-(1-m^{-1})/N} \tag{4.1}$$

which differs considerably from the limiting value m^{-1} for small particle numbers. For $N=2$ these are the $L=0$ states which exist whenever q is half integer. For $N=3$ such states exist whenever q is an odd integer, and they are then exact eigenstates of the Hamiltonian, because they are the only rotational symmetric three-particle states.

The spin-zero three-particle states can be constructed as

$$|\Psi^{(3)}_{LJ}\rangle = C\sum_m (-1)^m a^\dagger_{-m} A^\dagger_{q,m}|0\rangle = \bar{C}\sum_{mnr}\begin{pmatrix} q & q & q \\ m & n & r\end{pmatrix} a^\dagger_m a^\dagger_n a^\dagger_r \tag{4.2}$$

where C, $\bar{C}$ are normalisation factors. To calculate the energy of this state the following commutators are useful:

$$\begin{aligned}
[a_{-n}, A^\dagger_{L,M}] &= \theta_L \sum_\alpha (-1)^M \sqrt{2L+1}\begin{pmatrix} q & q & L \\ -n & \alpha & -M\end{pmatrix} a^\dagger_\alpha \\
[a^\dagger_{-n}, A_{L,M}] &= -\theta_L \sum_\alpha (-1)^M \sqrt{2L+1}\begin{pmatrix} q & q & L \\ -n & \alpha & -M\end{pmatrix} a_\alpha
\end{aligned} \tag{4.3}$$

where $\theta_L = 1-(-1)^{2q+L}$. Utilising these commutation rules, one finds

$$\begin{aligned}
\langle \Psi^{(3)}_{LJ}|\Psi^{(3)}_{LJ}\rangle &= 6(2q+1)C^2 \\
\langle \Psi^{(3)}_{LJ}|W_J|\Psi^{(3)}_{LJ}\rangle &= -36(2q+1)C^2\delta_{J,q}
\end{aligned} \tag{4.4}$$

which leads to an energy which is precisely the sum of three equal pair-energies,

$$E^{(3)}_0 = -6\frac{\beta_q}{\sqrt{2q+1}} = 3E^{(2)}_q. \tag{4.5}$$

The value of the pair energy corresponds to an angle $\vartheta_q = \cos^{-1}(-\frac{1}{2}) = 2\pi/3$ between the electrons. This is the appropriate value for the three electrons to be as far apart as possible, being evenly placed around a great circle. Such a simple relationship cannot be expected to hold for general N, but it can be used as a starting point for an estimation of energies for up to $N=32$ electrons [28].

5. The four-particle spin-zero states

We may also use the two-particle operators (2.9) as building blocks when constructing states with higher (even) particle numbers. In particular the four-particle spin zero states can be constructed as

$$|J\rangle = \sum_{J_z=-J}^{J} (-1)^{J_z} A^\dagger_{J,J_z} A^\dagger_{J,-J_z}|0\rangle. \tag{5.1}$$

These states are not all linearly independent, because there exist only of order $q/3$ linearly independent four-particle states with zero angular momentum [22], while there are q non-zero states $|J\rangle$. This fact is reflected in the large degeneracy of their scalar products (calculated in appendix 3),

$$\langle L|J\rangle = 2\theta_L\theta_J\sqrt{(2L+1)(2J+1)}[1+2(-1)^L R]_{LJ} \tag{5.2}$$

where R is the matrix defined in equation (2.13). Due to the θ factors it is at this point convenient to split $R_{L,J}$ into an odd and an even part (depending upon whether the indices L, J are odd or even),

$$\boldsymbol{R} = \begin{pmatrix} R_{++} & R_{+-} \\ R_{-+} & R_{--} \end{pmatrix}. \tag{5.3}$$

From sum rules for the 6-j symbols [26] it follows that

$$\boldsymbol{R}^2 = 1 \qquad \boldsymbol{RPR} = \boldsymbol{PRP} \tag{5.4}$$

where $\boldsymbol{P}$ is the parity matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. These relations imply that

$$(1-R_{++})(1+2R_{++}) = 0 \qquad (1+R_{--})(1-2R_{--}) = 0 \tag{5.5}$$

from which it can be read that the operators

$$P = \tfrac{1}{3}(1+2R_{++}) \qquad Q = \tfrac{2}{3}(1-R_{++}) \tag{5.6}$$

respectively

$$\bar{P} = \tfrac{1}{3}(1-2R_{--}) \qquad \bar{Q} = \tfrac{2}{3}(1+R_{--}) \tag{5.7}$$

are projection operators (in the even, respectively odd, parity subspace) onto the subspaces corresponding to eigenvalues 1 and $-\frac{1}{2}$, respectively -1 and $\frac{1}{2}$.

To construct an orthonormal basis of spin zero states we first (when $2q$ is odd) find an orthonormal set $\{\chi_L^{(\alpha)}\}$ of eigenvalue-1 eigenvectors for R_{++}, or (when $2q$ is even) find an orthonormal set $\{\chi_L^{(\alpha)}\}$ of eigenvalue -1 eigenvectors for R_{--}. The states

$$|\alpha\rangle = \sum_L \frac{1}{\sqrt{24(2L+1)}} \chi_L^{(\alpha)} |L\rangle \tag{5.8}$$

then form an orthonormal basis of $J^2 = 0$ states. The number D of these states are [22] $D = (k, k-1, k, k, k, k)$ when $2q = (6k-3, 6k-2, 6k-1, 6k, 6k+1, 6k+2)$.

To generate the basis vectors $\chi_L^{(\alpha)}$ we may utilise that (when $2q$ is odd)

$$P_{LJ} = \sum_\alpha \chi_L^{(\alpha)} \chi_J^{(\alpha)}$$

with the matrix indices restricted to be even, $L, J = 2q-1, 2q-3, \ldots$. This implies that $\chi_L^{(1)} \equiv P_{2q-1,L}/\sqrt{P_{2q-1,2q-1}}$ is a normalised eigenvector, and that $P_{LJ}^{(2)} \equiv P_{LJ} - \chi_L^{(1)}\chi_J^{(1)}$ is a projection operator onto the subspace orthogonal to $\chi^{(1)}$. Furthermore, the matrix elements $P_{LJ}^{(2)}$ are non-zero only when $L, J = 2q-3, 2q-5, \ldots$. Thus, we may define $\chi_L^{(2)} \equiv P_{2q-3,L}/\sqrt{P_{2q-3,2q-3}}$, and construct the next projection operator $P_{LJ}^{(3)} \equiv P_{LJ}^{(2)} - \chi_L^{(2)}\chi_J^{(2)}$. By repetition of this procedure a complete orthonormal basis is constructed. The last basis vector generated in this way, $\chi^{(D)}$, will correspond to the four-particle Laughlin state,

$$|\Psi_{L-J}^{(4)}\rangle = \sum_L \frac{1}{\sqrt{24(2L+1)}} \chi_L^{(D)} |L\rangle \tag{5.9}$$

when $2q = 6k-3$.

We further need to calculate the matrix elements of the interaction operator (2.5). In appendix 3 we find

$$-\sqrt{2K+1}\langle L|W_K|J\rangle = 4\theta_L\theta_J\theta_K\sqrt{(2L+1)(2J+1)}[1+2(-1)^K R]_{LK}[1+2(-1)^K R]_{KJ}$$

from which it follows that

$$-\sqrt{2K+1}\langle\alpha|W_K|\beta\rangle = 6\theta_K \chi_K^{(\alpha)} \chi_K^{(\beta)}.$$

That is

$$\langle\alpha|V|\beta\rangle = -12 \sum_K{}' \frac{\beta_K}{\sqrt{2K+1}} \chi_K^{(\alpha)} \chi_K^{(\beta)} \tag{5.10}$$

where the sum is restricted to such K that $2q+K$ is odd.

Explicit calculation of the coefficients β_K and the basis vectors $\chi^{(\alpha)}$, and the diagonalisation of the $D \times D$ Hamiltonian matrix now has to be done numerically. This we discuss in the next section.

6. Numerical results and discussion

Since we have made use of the exact properties of the matrix R_{LJ} when constructing the projection operators P_{LJ}, $\bar{P}_{LJ}$, from which the basis vectors and the Hamiltonian matrix is constructed, the numerical calculation of this matrix is the most crucial part of the computation. As a test of the accuracy of the projection property we have computed

$$\sigma_\infty(q) \equiv \|P^2 - P\|_\infty \equiv \max_{LJ} |[P^2 - P]_{LJ}|$$

and (6.1)

$$\sigma_2(q)^2 \equiv \|P^2 - P\|_2 \equiv \sum_{LJ} \{[P^2 - P]_{LJ}\}^2$$

and the similar quantities with $P \to \bar{P}$. Computing the R_{LJ} in an essentially straightforward way, by evaluating the explicit summation formula for the 6-j symbols, both $\sigma_\infty(q)$ and $\sigma_2(q)$ grow exponentially with q, approximately like $\exp(q/3)$ with a prefactor of about 10^{-17}. This limits q to about 85 [29].

The algorithm for constructing basis vectors can then be tested by computing

$$\delta_\infty(q) \equiv \max_{LJ} \left| P_{LJ} - \sum_\alpha \chi_L^{(\alpha)} \chi_J^{(\alpha)} \right| \qquad \text{and} \qquad \delta_2(q)^2 \equiv \sum_{LJ} \left\{ P_{LJ} - \sum_\alpha \chi_L^{(\alpha)} \chi_J^{(\alpha)} \right\}^2 \tag{6.2}$$

and the similar quantities with $P \to \bar{P}$. We have found that the size of these quantities are essentially proportional to the $\sigma(q)$, which indicates that our algorithm for constructing basis vectors from P or $\bar{P}$ is numerically stable. Finally, diagonalisation of the Hamiltonian matrix is done by a standard numerical package. Since the dimension D of our matrices are quite low ($D \leqslant 28$ for $q \leqslant 85$) this step does not seem to pose any numerical challenge.

Further, as an overall check the numerics may be tested against some exactly solvable models.

(*a*) $v_L = \delta_{L,0}$. The corresponding values of α_L and β_J are

$$\alpha_L = \tfrac{1}{2}(2q+1)\delta_{L,0} \qquad \frac{\beta_J}{\sqrt{2J+1}} = \tfrac{1}{2}(2q+1) \begin{Bmatrix} q & q & J \\ q & q & 0 \end{Bmatrix} = -\tfrac{1}{2}. \tag{6.3}$$

This case corresponds to a constant pair-interaction potential. Thus the interaction is essentially the square of the particle number operator $\boldsymbol{N}$.

$$V = \tfrac{1}{2} : \boldsymbol{N}^2 : = \tfrac{1}{2}(\boldsymbol{N}^2 - \boldsymbol{N}). \tag{6.4}$$

(*b*) $v_L = \delta_{L,1}$. In this case we have that

$$\alpha_L = \frac{1}{2\sqrt{3}}(2q+1)\frac{q}{q+1}\delta_{L,1}$$
$$\frac{\beta_J}{\sqrt{2J+1}} = \tfrac{1}{2}(2q+1)\frac{q}{q+1}\begin{Bmatrix} q & q & J \\ q & q & 1 \end{Bmatrix} = -\frac{1}{4(q+1)^2}[J(J+1)-2q(q+1)] \tag{6.5}$$

which corresponds to a pair interaction $V = \cos\vartheta$. In appropriate combination with the constant potential (6.3) this may be considered as a spherical analogue of the harmonic pair interaction. The interaction can be written

$$V = \frac{1}{2(q+1)^2}[\boldsymbol{J}^2 - q(q+1)\boldsymbol{N}] \tag{6.6}$$

where $\boldsymbol{J}$ is the angular momentum operator, from which the spectrum can be read out immediately.

(*c*) $v_L = 2L+1$. This corresponds to $V(\Omega,\Omega') \propto \delta(\Omega-\Omega')$. Due to the Pauli principle two electrons of the same spin polarisation cannot be in the same position. Thus, this interaction belongs to the class of irrelevant potentials, $V \equiv 0$.

With the Coulomb interaction we choose to measure energy in units of $e^2/4\pi\epsilon l_B$. Analytic results are available at $q = \frac{3}{2}$ (since this corresponds to a filled Landau level, $\nu = 1$),

$$\varepsilon(1) = -\frac{2^{4q-1}}{\sqrt{q}}\binom{4q+1}{2q}^{-1} = -\frac{32}{35}\sqrt{\frac{2}{3}} \tag{6.7}$$

and at $q = \frac{5}{2}$ (since this is related to the two-particle case by particle-hole symmetry†),

$$\varepsilon(2/3) = -\frac{212}{231}\sqrt{\frac{2}{5}}. \tag{6.8}$$

Also, the $q = 3$ case can be compared (numerically) with the three-particle Laughlin state, again by use of particle-hole symmetry. The particle-hole symmetry relation is

$$\nu[\varepsilon(\nu)-\varepsilon(1)] = (1-\nu)[\varepsilon(1-\nu)-\varepsilon(1)] \tag{6.9}$$

as can be deduced by rewriting the Hamiltonian (2.15) in terms of hole creation and annihilation operators. To avoid finite-size corrections it is essential that all energies are evaluated at the same value of q, and that the correct definition (4.1) of filling factor is used.

In table 1 we summarise the numerical results for the lowest energy $J = 0$, $N = 2$, 3 and 4 states, as well as their charge-conjugated states, for $q \leqslant 9$. It is in perfect agreement with the relations imposed by particle-hole symmetry, the exactly calculable cases, and with the $N = 3$, $q = 3$, 5 and $N = 4$, $q = \frac{9}{2}$, $\frac{15}{2}$ results of Fano *et al* to all decimals given in their paper [21].

In table 2 we list the energies of the lowest $J = 0$, $N = 4$ states, as well as the energy gaps to the first excited $J = 0$, $N = 4$ states. The q values chosen belong to the set for which a spherical analogue of a Laughlin state, $\Psi^{(4)}_{\square}$, exists. In the second and third columns we list the linear dimensions of the eigenvalue problems to be solved if one makes a full symmetry reduction as in this paper ($D(q)$), or works with the configuration interaction matrices ($C(q)$). As can be seen, the difference is quite formidable, in

† The energy per particle is $\varepsilon = -(6q+1)/[(8q+2)\sqrt{q}]$ for the $J = 0$, $N = 2$ state.

Table 1. Energies of the lowest $J = 0$ states for some low values of particle numbers N and magnetic charge q.

q	$N=2$	$N=3$	$N=4$	$N=2q-3$	$N=2q-2$	$N=2q-1$
1.5	−0.583 211 843 520	—	−0.746 511 159 705	—	—	−0.583 211 843 520
2.5	−0.459 967 659 661	—	−0.580 435 380 048	−0.459 967 659 661	—	−0.580 435 380 048
3.0	—	−0.503 779 605 517	−0.550 097 721 518	−0.503 779 605 517	−0.550 097 721 518	—
3.5	−0.391 983 154 805	—	−0.488 144 594 138	−0.488 144 594 138	—	−0.584 306 033 471
4.0	—	—	−0.457 362 103 138	−0.500 681 949 067	—	—
4.5	−0.347 350 699 530	—	−0.475 024 438 546	−0.539 636 235 601	—	−0.588 482 547 166
5.0	—	−0.398 570 460 550	−0.394 574 759 254	−0.510 365 072 269	−0.564 934 020 386	—
5.5	−0.315 166 276 352	—	−0.425 233 561 501	−0.543 303 552 303	—	−0.592 132 089 754
6.0	—	—	−0.416 940 544 852	−0.551 165 595 968	—	—
6.5	−0.290 542 422 427	—	−0.389 325 543 639	−0.549 416 553 023	—	−0.595 210 314 248
7.0	—	−0.340 078 986 641	−0.379 210 484 728	−0.554 126 621 043	−0.575 578 770 864	—
7.5	−0.270 916 533 820	—	−0.377 578 193 087	−0.560 715 350 577	—	−0.597 802 872 822
8.0	—	—	−0.349 705 811 749	−0.558 093 132 840	—	—
8.5	−0.254 797 897 926	—	−0.350 533 567 472	−0.563 950 847 163	—	−0.600 002 776 401
9.0	—	−0.301 551 265 296	−0.347 071 233 380	−0.567 809 619 618	−0.583 104 337 753	—

Table 2. Energies per particle $\varepsilon(q)$ of the lowest lying $J=0$, $N=4$ states, and the gap $\Delta(q)$ to the first excited $J=0$ state, for some q values at which the spherical analogues of the Laughlin states exist. The dimensions of the matrix eigenvalue problems when making a full symmetry reduction ($D(q)$), or only block diagonalising J_z ($C(q)$) are also shown.

q	$D(q)$	$C(q)$	$\varepsilon(q)$	$\Delta(q)$
10.5	4	241	−0.323 084 2580	0.010 789 2038
16.5	6	956	−0.260 758 1027	0.006 739 0283
22.5	8	2 451	−0.224 533 6079	0.004 540 3806
28.5	10	5 014	−0.200 143 0331	0.003 251 9259
34.5	12	8 933	−0.182 289 5696	0.002 467 2199
40.5	14	14 496	−0.168 493 7978	0.001 952 3249
46.5	16	21 991	−0.157 420 1196	0.001 593 8741
52.5	18	31 706	−0.148 277 0491	0.001 332 7972
58.5	20	43 929	−0.140 561 832	0.001 135 802
64.5	22	58 948	−0.133 937 76	0.000 982 88
70.5	24	77 051	−0.128 169 8	0.000 861 4
76.5	26	98 526	−0.123 088	0.000 763
82.5	28	123 661	−0.118 56	0.000 67

particular when taking into account that the diagonalisation time grows like the third power of the matrix dimension. However, the region of parameter space covered is not of the highest physical interest, and the main purpose of the data in table 2 is to provide accurate answers to a well defined standard problem, on which other numerical algorithms (e.g. the Lanczos method) can be tested.

In figure 2 we show the strong correlation between the energy for the lowest $J=0$ state, E/N, and the total number $D(4, 0; q)$ of such states. Each time q increases by 3 a new $J=0$ state is introduced, and this appears to become the new ground state (cf figure 3). This new state is introduced at q values of $\frac{3}{2}, \frac{9}{2}, \ldots$, i.e. at precisely those q values where the spherical analogues of the Laughlin states exist. The stability of these particular filling factors is further enhanced because the new state temporarily disappears when q is increased by $\frac{1}{2}$. Note that there also are cusps in E/N at $q=3$, $6, \ldots$. This is the sequence predicted for the first hierarchical levels, $q=\frac{1}{2}m(N-1)\pm\frac{1}{2}(1+N/p)$, with $p=2$ and m an odd integer (and $N=4$). In figure 3 we show the energy levels appear to lie on fairly smooth trajectories as q is varied, with the cusps that appear being due to the occurrence of new trajectories.

In figure 4 we have plotted the energy gap between the lowest and the first excited $J=0$ states. Perhaps contrary to expectation this gap is not the largest at the particular fractions $q=\frac{9}{2}, \frac{15}{2}, \ldots$. The reason for this can be read from figure 3. As a new state is introduced and becomes the new ground state, the previous lowest energy $J=0$ state becomes the new first ($J=0$) excited state. In summary, the main content of this paper is as follows.

(i) We have studied the symmetry reduction of a standard class of FQHE Hamiltonians defined on the 2-sphere, with respect to the full rotation group, and we have explicitly considered the simplest non-trivial case. As we have shown the analytic work can be carried out quite neatly and completely in this case. The remaining numerical work was easily done on a personal computer. In comparison we think the parameter range we have covered by this method would be very difficult and time consuming to handle even on the largest supercomputers without the symmetry reduction. An additional advantage is that the most time-consuming part of our computation,

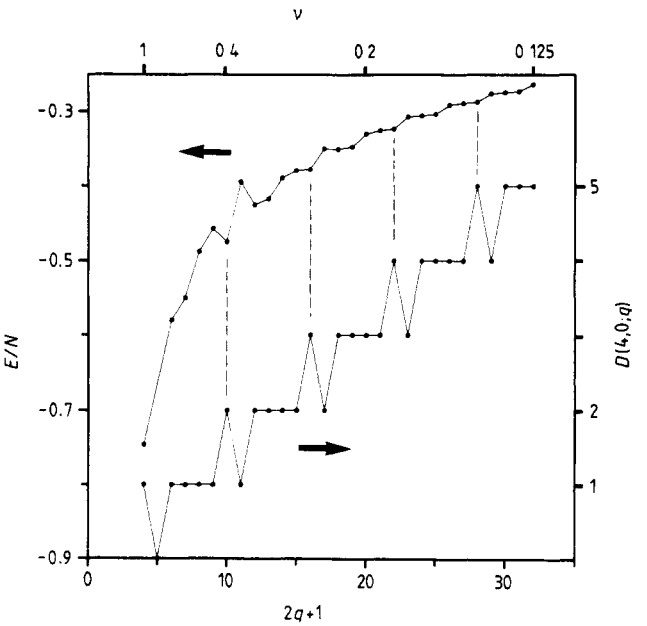

Figure 2. The correlation between the lowest $J = 0$ energy per particle, E/N, and the total number of $J = 0$ states, $D(4, 0; q)$. The indicated q values at which a new state first appears correspond to the spherical version of the Laughlin sequence, $q = \frac{1}{2}m(N-1)$, with m an odd integer. The additional cusps which appear in E/N at $q = 3, 6, \ldots$ fit with the sequence for the first hierarchical levels, $q = \frac{1}{2}m(N-1) \pm \frac{1}{2}(1 + N/p)$, with $p = 2$ and m an odd integer.

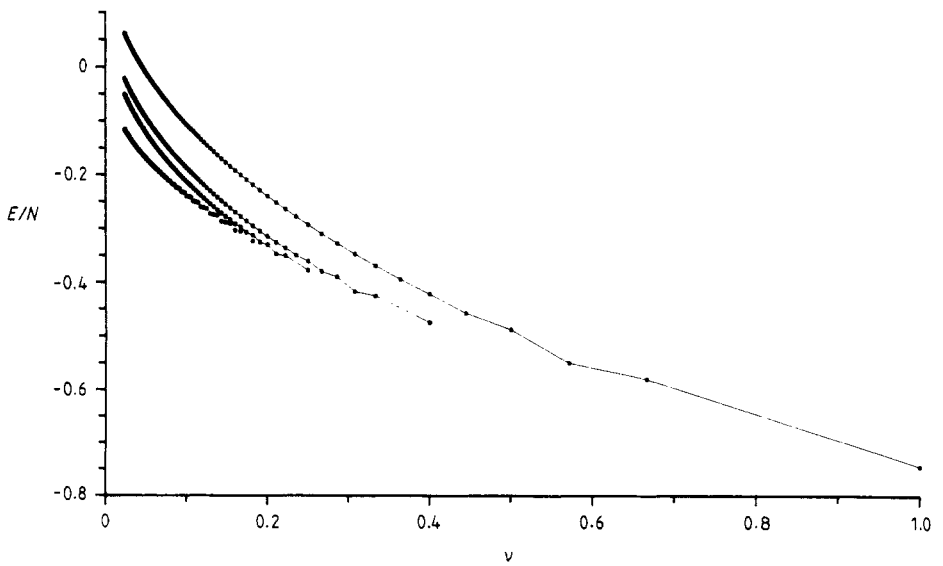

Figure 3. The energies of the lowest and the three highest $J = 0$ states, for four particles on the sphere, as function of the filling factor. The energy levels appear to lie on fairly smooth trajectories as q is varied.

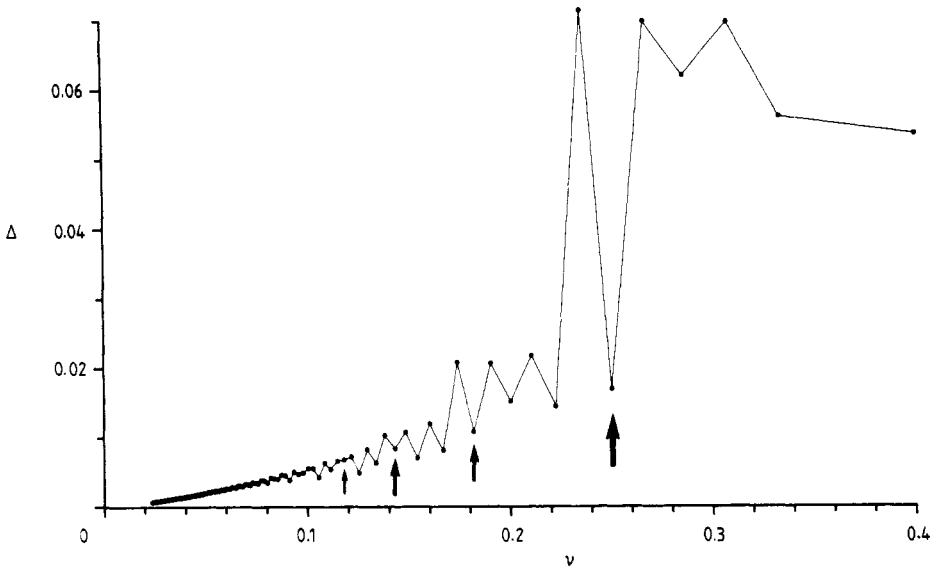

Figure 4. The energy gap between the lowest and the first excited $J = 0$, $N = 4$ states. Some particular fillings at which a Laughlin state occurs are indicated with arrows.

which is the numerical calculation of the matrix R_{LJ} and the basis vectors $\chi_L^{(\alpha)}$, only has to be done once (for each value of q), regardless of the type of interaction potential.

(ii) We have provided accurate answers to a standard case problem, against which numerical algorithms not involving symmetry reduction can be tested in a non-trivial way.

(iii) Somewhat to our surprise there is, even at the very low particle number considered by us, clear evidence for the existence of particularly stable states when the magnetic monopole charge $q = \frac{9}{2}, \frac{15}{2}, \ldots$.

Appendix 1. The transformation matrix

We have defined

$$V_L = \frac{1}{\sqrt{2L+1}} \sum_{M=-L}^{L} (-1)^M :C_{L,M} C_{L,-M}: . \tag{A1.1}$$

Inserting $C_{L,M}$ from (2.7) we get

$$V_L = \sqrt{2L+1} \sum_{Mmnm'n'} (-1)^{M+2q-n-n'} \begin{pmatrix} q & q & L \\ m & -n & -M \end{pmatrix} \begin{pmatrix} q & q & L \\ m' & -n' & M \end{pmatrix} :a_m^+ a_n a_{m'}^+ a_{n'}: . \tag{A1.2}$$

Normal ordering the fermion operators to the form $a_m^+ a_{m'}^+ a_n a_{n'}$ yields a minus sign. Further, inverting the definitions (2.9) by use of the orthogonality relation for the Wigner 3-j symbols [26]

$$\sum_{LM} (2L+1) \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} \begin{pmatrix} q & q & L \\ r & s & -M \end{pmatrix} = \delta_{mr} \delta_{ns} \tag{A1.3}$$

the products $a^\dagger_m a^\dagger_{m'}$, respectively $a_n a_{n'}$, can be expressed in terms of $A^\dagger_{J,J_z}$, respectively A_{J',J'_z},

$$\begin{aligned} a_m a_n &= -\sum_{LM} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} A_{LM} \\ a^\dagger_m a^\dagger_n &= \sum_{LM} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} A^\dagger_{LM}. \end{aligned} \tag{A1.4}$$

This leads to the expression

$$\begin{aligned} V_L = \sqrt{2L+1} \sum_{J,J_z,J',J'_z} (-1)^{J_z+J'_z} \sqrt{(2J+1)(2J'+1)}\, A^\dagger_{J,J_z} A_{J',J'_z} \\ \times \sum_{Mmnm'n'} (-1)^{M+2q-n-n'} \begin{pmatrix} q & q & J \\ m' & m & -J_z \end{pmatrix} \begin{pmatrix} q & L & q \\ -m & M & n \end{pmatrix} \\ \times \begin{pmatrix} L & q & q \\ -M & -m' & n' \end{pmatrix} \begin{pmatrix} q & q & J' \\ n & n' & -J'_z \end{pmatrix}. \end{aligned} \tag{A1.5}$$

The last sum over the four 3-j symbols gives the factor [26]

$$\frac{-\delta_{J_z,J'_z}\delta_{J,J'}}{2J+1} \begin{Bmatrix} q & q & L \\ q & q & J \end{Bmatrix}.$$

From this it follows that $V_L = \Sigma_J R_{LJ} W_J$, with

$$R_{LJ} = \sqrt{(2L+1)(2J+1)} \begin{Bmatrix} q & q & L \\ q & q & J \end{Bmatrix} \tag{A1.6}$$

and W_J as defined in (2.11).

Appendix 2. The commutator algebra

We have defined the two-fermion operators

$$\begin{aligned} A_{L,M} &= -\sum_{mn} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a_m a_n \\ A^\dagger_{L,M} &= \sum_{mn} (-1)^M \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix} a^\dagger_m a^\dagger_n \\ C_{L,M} &= \sum_{mn} (-1)^{M-n+q} \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & -n & -M \end{pmatrix} a^\dagger_m a_n. \end{aligned} \tag{A2.1}$$

The operators $(-1)^M A_{L,-M}$, $A^\dagger_{LM}$ and C_{LM}, $M = -L, \ldots, L$ all transform under rotations as the components of a spin-L multiplet. Note that A_{LM} and $A^\dagger_{LM}$ are identically zero when $(-1)^{2q+L} = 1$, due to the symmetry properties of the 3-j symbols. These relations may be inverted to (A1.4), and

$$a^\dagger_m a_n = \sum_{LM} (-1)^{M+n-q} \sqrt{2L+1} \begin{pmatrix} q & q & L \\ m & -n & -M \end{pmatrix} C_{LM}. \tag{A2.2}$$

To calculate the commutators we first find directly from (A2.1)

$$[A_{LM}, A^{\dagger}_{JN}] = \sum_{mnrs} (-1)^{M+N}\sqrt{(2L+1)(2J+1)} \times \begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix}\begin{pmatrix} q & q & J \\ r & s & -N \end{pmatrix}[a^{\dagger}_r a^{\dagger}_s, a_m a_n]$$

next we use the canonical anticommutation relations to obtain

$$[a^{\dagger}_r a^{\dagger}_s, a_m a_n] = \delta_{ms} a^{\dagger}_r a_n - \delta_{ns} a^{\dagger}_r a_m + \delta_{nr} a^{\dagger}_s a_m - \delta_{mr} a^{\dagger}_s a_n + \delta_{mr}\delta_{ns} - \delta_{ms}\delta_{nr}$$

and use the relations (A2.2) to express the $a^{\dagger}a$ in terms of C_{KR}. We are then left with sums over products of two or three 3-j symbols, of the form

$$\sum_{mn} (2L+1)\begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix}\begin{pmatrix} q & q & J \\ m & n & -N \end{pmatrix} = \delta_{LJ}\delta_{MN}$$

and

$$\sum_{mnrs} (-1)^{M+N+R+n-q}\begin{pmatrix} q & q & L \\ m & n & -M \end{pmatrix}\begin{pmatrix} q & q & J \\ r & m & -N \end{pmatrix}\begin{pmatrix} q & q & K \\ r & -n & -R \end{pmatrix} = (-1)^{L+N}\begin{Bmatrix} L & J & K \\ q & q & q \end{Bmatrix}\begin{pmatrix} L & J & K \\ -M & N & -R \end{pmatrix}$$

with the object in curly braces a Wigner 6-j symbol. Collecting terms, and repeating the above procedure for the other commutators as well, we obtain

$$\begin{aligned}
[A_{LM}, A^{\dagger}_{JN}] &= \theta_L\theta_J\left\{\tfrac{1}{2}\delta_{LJ}\delta_{MN} + \sum_{KR} (-1)^{L+N} f_{LJK}\begin{pmatrix} L & J & K \\ -M & N & -R \end{pmatrix} C_{KR}\right\} \\
[C_{LM}, A_{JN}] &= \theta_J \sum_{KR} (-1)^{L+K+N} f_{LJK}\begin{pmatrix} L & J & K \\ -M & N & -R \end{pmatrix} A_{KR} \\
[C_{LM}, A^{\dagger}_{JN}] &= -\theta_J \sum_{KR} (-1)^{L+K+R} f_{LJK}\begin{pmatrix} L & J & K \\ M & N & -R \end{pmatrix} A^{\dagger}_{KR} \\
[C_{LM}, C_{JN}] &= -\sum_{KR} [1-(-1)^{L+J+K}](-1)^{R} f_{LJK}\begin{pmatrix} L & J & K \\ M & N & -R \end{pmatrix} C_{KR}
\end{aligned} \tag{A2.3}$$

where $\theta_L = [1-(-1)^{2q+L}]$, and

$$f_{LJK} = \sqrt{(2L+1)(2J+1)(2K+1)}\begin{Bmatrix} L & J & K \\ q & q & q \end{Bmatrix}.$$

The Lie algebra (A2.3) is just a particular way of writing the so($4q+2$) algebra of the bilinears in a_n and $a^{\dagger}_n$. It contains many subalgebras. Of particular interest is the su(2) quasispin algebra which exists whenever q is half integer. We have made a quasispin analysis of the class of models studied in this paper [27].

Appendix 3. Four-particle matrix elements

We have defined the four-particle spin-zero states

$$|J\rangle = \sum_{N=-J}^{J} (-1)^{N} A^{\dagger}_{JN} A^{\dagger}_{J,-N}|0\rangle \tag{A3.1}$$

where $|0\rangle$ is the zero-particle state. To evaluate the scalar product

$$\langle L|J\rangle = \sum_{MN} (-1)^{M+N} \langle 0|A_{L,-M} A_{LM} A^{\dagger}_{JN} A^{\dagger}_{J,-N}|0\rangle \tag{A3.2}$$

we first commute the rightmost annihilation operator A_{LM} to the right, using the commutator algebra (A2.3). That is, we write $A_{LM}A^{\dagger}_{JN} = A^{\dagger}_{JN}A_{LM} + [A_{LM}, A^{\dagger}_{JN}]$, which gives two terms.

In the first term we may replace $A_{LM}A^{\dagger}_{J,-N}$ with their commutator, since A_{LM} gives zero when operating on $|0\rangle$. Furthermore, this commutator may be replaced by $\frac{1}{2}\theta_L\theta_J\delta_{LJ}\delta_{M,-N}$, since the remainder contains a C_{KR} which also gives zero when operating on $|0\rangle$. Thus, the first term leads to the expression

$$\sum_{MN} (-1)^{M+N} \tfrac{1}{2}\theta_L\theta_J\delta_{LJ}\delta_{M,-N} \langle 0|A_{L,-M}A^{\dagger}_{JN}|0\rangle = (2L+1)\theta_L\theta_J\delta_{LJ}. \tag{A3.3}$$

Now we turn to the second term. This becomes

$$\langle 0|A_{L,-M}[A_{LM}, A^{\dagger}_{JN}]A^{\dagger}_{J,-N}|0\rangle = \tfrac{1}{2}\theta_L\theta_J\delta_{LJ}\delta_{MN}\langle 0|A_{L,-M}A^{\dagger}_{J,-N}|0\rangle$$
$$+ \theta_L\theta_J \sum_{KR} (-1)^{L+N} f_{LJK} \begin{pmatrix} L & J & K \\ -M & N & -R \end{pmatrix} \langle 0|A_{L,-M}C_{KR}A^{\dagger}_{J,-N}|0\rangle$$

when inserting the commutator from (A2.3). In the second of these two terms we may replace $C_{KR}A^{\dagger}_{J,-N}$ with their commutator, since C_{KR} gives zero when operating on $|0\rangle$. This gives a contribution

$$\sum_{MN} (-1)^{M+N}\theta_L\theta_J \sum_{KR} (-1)^{L+N} f_{LJK} \begin{pmatrix} L & J & K \\ -M & N & -R \end{pmatrix}$$
$$\times -\theta_J \sum_{HS} (-1)^{K+H+S} f_{KJH} \begin{pmatrix} K & J & H \\ R & -N & -S \end{pmatrix} \langle 0|A_{L,-M}A^{\dagger}_{HS}|0\rangle$$
$$= \theta_{LJ} \sum_{K} (-1)^{2q+K} (2K+1) \begin{Bmatrix} L & J & K \\ q & q & q \end{Bmatrix} \begin{Bmatrix} J & L & K \\ q & q & q \end{Bmatrix} = \theta_{LJ} \begin{Bmatrix} q & q & L \\ q & q & J \end{Bmatrix}$$

where in the last step we have used a standard sum rule for 6-j symbols [26] and $\theta_{LJ} = 4\theta_L\theta_J(2L+1)(2J+1)(-1)^L$. Collecting the three contributions, we get

$$\langle L|J\rangle = 2\theta_L\theta_J\sqrt{(2L+1)(2J+1)}\,[1+2(-1)^L R]_{LJ}. \tag{A3.4}$$

Now we turn to the matrix elements

$$-\sqrt{2K+1}\langle L|\,W_K\,|J\rangle = \sum_{MNR} (-1)^{M+N} \langle 0|A_{L,-M}A_{LM}A^{\dagger}_{KR}A_{KR}A^{\dagger}_{JN}A^{\dagger}_{J,-N}|0\rangle \tag{A3.5}$$

which we evaluate by the same technique as above. We first commute the operator A_{KR} two places to the right, obtaining three terms:

$$\sum_{MNR} (-1)^{M+N}\theta_J\theta_K \Big\{ \tfrac{1}{2}\delta_{KJ}\delta_{RN}\langle 0|A_{L,-M}A_{LM}A^{\dagger}_{KR}A^{\dagger}_{J,-N}|0\rangle + (N \leftrightarrow -N)$$
$$+ \sum_{HS} (-1)^{K+N} f_{KJH} \begin{pmatrix} K & J & H \\ -R & N & -S \end{pmatrix} \langle 0|A_{L,-M}A_{LM}A^{\dagger}_{KR}C_{HS}A^{\dagger}_{J,-N}|0\rangle \Big\}.$$

The first two of these are similar to the expression in (A3.2), and lead to a contribution $4\theta_L\theta_J\theta_K[(2L+1)(2J+1)]^{1/2}[1+2(-1)^K R]_{LJ}\delta_{JK}$. In the last term we may replace $C_{HS}A^\dagger_{J,-N}$ by their commutator, obtaining a contribution

$$\sum_{MNRHS}(-1)^{M+K}\theta_J\theta_K f_{KJH}\begin{pmatrix} K & J & H \\ -R & N & -S\end{pmatrix}$$
$$\times -\sum_{GT}\theta_J(-1)^{H+G+T}f_{HJG}\begin{pmatrix} H & J & G \\ S & -N & -T\end{pmatrix}\langle 0|A_{L,-M}A_{LM}A^\dagger_{KR}A^\dagger_{GT}|0\rangle.$$

Due to rotation symmetry the matrix element which remains to be evaluated is non-zero only when $A^\dagger_{KR}A^\dagger_{GT}$ can be coupled to an angular momentum zero operator. This requires $G=K$ and $T=-R$, and leads to the same matrix element as in (A3.2), e.g.

$$\sum_{MT}(-1)^{M+T}\langle 0|A_{L,-M}A_{LM}A^\dagger_{KR}A^\dagger_{GT}|0\rangle$$
$$=2\theta_L\theta_K\sqrt{(2L+1)(2K+1)}\,[1+2(-1)^K R]_{LK}\delta_{KG}\delta_{R,-T}.$$

The sum over the remaining factors is independent of this expression, and gives a factor $(-1)^K 2\theta_J\theta_K[(2J+1)/(2K+1)]^{1/2}R_{KJ}$. Thus, the last term becomes $8(-1)^K\theta_L\theta_J\theta_K[(2L+1)(2J+1)]^{1/2}[1+2(-1)^K R]_{LK}R_{KJ}$. Collecting all terms we finally get

$$-\sqrt{2K+1}\,\langle L|W_K|J\rangle$$
$$=4\theta_L\theta_J\theta_K\sqrt{(2L+1)(2J+1)}\,[1+2(-1)^K R]_{LK}[1+2(-1)^K R]_{KJ}. \qquad \text{(A3.6)}$$

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